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Vector phase measurement in multipath quantum interferometry

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Abstract. We introduce vector phase states for multipath quantum interferometry and construct the vector phase positive operator-valued measure. We calculate $SU(3)$ phase distributions for three-path quantum interferometry and discuss measurement limits.

1. Introduction

Precise interferometric measurements of phase shifts are important for many applications, yet complementarity between particle number and phase limits the amount of information which can be extracted from the interferometer [1]. These limits are well understood in the context of two-path interferometry, but the development of multipath quantum interferometers (MQI) [2] raises issues about the bounds to estimating simultaneous multiple phase shifts [3, 4]. Our aim is to establish rigorous bounds on estimating this multiple phase shift. Specifically, we (1) employ the $SU(N)$ group to describe the interferometer and identify the Fock basis for the input state with the (Cartan) weight basis, (2) develop the $SU(N)$ 'vector phase' state (VPS) as the dual basis to the weight states [5], (3) present a class of MQI designs for which a 'rotated' VPS basis is translated by MQI, (4) determine 'vector phase' distributions for states which can be studied via parametric estimation theory, (5) establish the relation between bounds on vector phase measurement in connection with the Fisher information matrix [6], and (6) calculate and plot $SU(3)$ vector phase distributions.

Lie group theory provides the natural language for describing interferometry as a set of unitary transformations. For a single-mode field, it is sufficient to introduce the annihilation and creation operators, a and $a^\dagger$, plus the identity operator, which together span the Heisenberg–Weyl (HW) algebra. The Fock number states $\{|n\rangle\}$ are eigenstates of the unitary phase-shift operator

$$\exp(\mathrm{i}\phi a^\dagger a). \tag{1.1}$$

Whereas the Fock state is an eigenstate of the phase-shift operator (1.1), the unnormalized phase state [1, 7]

$$|\theta\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n\theta}|n\rangle \tag{1.2}$$

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is translated to $|\theta + \phi\rangle$ by this unitary phase-shift operation. Consequently, the phase state basis is dual to the Fock basis, in the same sense that the position and momentum bases are dual, and the phase basis allows rigorous bounds to be established on the information which can be extracted by phase measurements [8].

Optimal extraction of phase-shift information corresponds to application of the infinitesimal positive-operator valued measure (POVM) [9] $\mathrm{d}E(\phi) = |\phi\rangle\langle\phi|\,\mathrm{d}\mu(\phi)$ with $\mathrm{d}\mu(\phi)$ a measure that guarantees normalization of the POVM and ensures that $\mathrm{d}E(\phi)$ is a resolution of the identity. Although not directly measurable in practice, the POVM has proven useful in establishing ultimate bounds to the information which can be extracted by any phase measurement.

Recognizing that passive, linear interferometry involves the mixing of at least two fields via $SU(2)$ optical elements [2,10], a (scalar) phase POVM for $SU(2)$ has been introduced [11,12]. In this approach, an arbitrary input state can be decomposed into a direct sum of states in distinct finite irreducible representations (irreps). In a lossless two-path quantum interferometer, particle number is conserved, and $SU(2)$ symmetry is preserved at each interferometer element [10].

Here we extend the $SU(2)$ analysis to analyse vector phase measurement for MQI, which is characterized by an $SU(N)$ symmetry [2]. We apply $SU(N)$ transformations to the study of $SU(3)$ quantum interferometry in general, and symmetric $SU(N)$ interferometry in particular. For $N = 3$, we calculate and plot $SU(3)$ vector phase distributions for certain states and study the scaling of the distribution widths with respect to the irrep parameter.

2. The interferometer transformation and representations

Lie group theory is applicable because we can treat the lossless system as a system which conserves particle number. This conservation law is a consequence of the $SU(N)$ symmetry of the interferometer, and this conservation principle applies at each of the optical elements. These elements can mix two fields together (labelled, for example, as k and l) or act on just one field to shift the phase. As particle number is conserved at each passive, linear, lossless optical element, it is convenient to introduce the Hermitian operators

$$\begin{aligned} M_k^k &= a_k^\dagger a_k \\ M_k^l &= a_k^\dagger a_l + a_l^\dagger a_k \\ M_l^k &= \mathrm{i}(a_k^\dagger a_l - a_l^\dagger a_k) \end{aligned} \tag{2.1}$$

for $k < l \leqslant N$, where $\{(a_k, a_k^\dagger)|k \in \mathbb{Z}_N\}$ are the field annihilation and creation operators.

The unitary transformation, corresponding to a passive, linear, lossless optical element (including a beam splitter, a mirror and/or a phase shifter), can thus be expressed as [10]

$$\mathcal{R}_k^l(\boldsymbol{\Omega}) = \exp\{\mathrm{i}\boldsymbol{\theta}\cdot(M_k^l, M_l^k, M_k^k - M_l^l)\}. \tag{2.2}$$

The 50/50 beam splitter corresponds to $\mathcal{R}_k^l(\pi/4, 0, 0)$ and the phase shifter to $\mathcal{R}_k^l(0, 0, \theta)$ (for the mirror $\theta = \pi$). An arbitrary $SU(N)$ transformation can be expressed as

$$\mathcal{I}(\Upsilon) = \exp\left(\mathrm{i}\sum_{k,l=1}^{N}\Upsilon_l^k M_k^l\right) \tag{2.3}$$

with real $N \times N$ matrix Υ. This transformation can be decomposed into a sequence of $SU(2)$ transformations [2, 13] indicating that a quantum interferometer consisting entirely of $SU(2)$ elements can realize an arbitrary $SU(N)$ quantum interferometer. The decomposition of the transformation (2.3) into a sequence of $SU(2)$ transformations is not unique.

The unitary operator (2.3) thus describes any passive, linear, lossless interferometer transformation of an input state into an output state. There are N^2-1 independent coefficients Υ_l^k in the exponent of (2.3) which are determined by parameter choices for the optical elements. We assume that the beam splitter reflectivities are completely known, and the unknown parameters, which will be estimated, are the $N-1$ phase shifts in each path of the quantum interferometer. These unknown phase shifts are represented by an $(N-1)$-dimensional 'vector phase' $\boldsymbol{\phi} = (\phi_i)$, with $0 \leqslant \phi_i < 2\pi$. That is, the vector phase is confined to a hypertoroidal domain $\mathcal{S}^1 \times \mathcal{S}^1 \times \cdots \times \mathcal{S}^1$.

Although there are $N-1$ distinct Casimir operators for $SU(N)$, the specification of the normalized particle number sum

$$S = N^{-1} \sum_{k=1}^{N} M_k^k \tag{2.4}$$

is sufficient to determine an irreducible representation as the N-field state consists solely of bosons: only the symmetric irreducible representation appears. The Cartan subalgebra of $SU(N)$ is spanned by $N-1$ linearly independent components of the operator $\boldsymbol{h}$

$$h_n = M_n^n - M_{n+1}^{n+1} \qquad 1 \leqslant n < N-1. \tag{2.5}$$

For a given irrep (determined by S), we introduce the orthonormal basis $\{|s\boldsymbol{m}\rangle\}$ such that

$$S|s\boldsymbol{m}\rangle = s|s\boldsymbol{m}\rangle \qquad \boldsymbol{h}|s\boldsymbol{m}\rangle = \boldsymbol{m}|s\boldsymbol{m}\rangle \tag{2.6}$$

with $\boldsymbol{m}$ the $(N-1)$-dimensional weight vector. The connection between the N-field Fock state $|\boldsymbol{n}\rangle$ and the weight basis is obtained by identifying

$$s = N^{-1} \sum_{\nu} n_\nu \qquad m_k = n_k - n_{k+1} \tag{2.7}$$

for $k \in \mathbb{Z}_{N-1}$. We now have a representation of N-field interferometry as an $SU(N)$ transformation with a bijective mapping between the Fock basis of N fields and the weight basis.

An arbitrary pure input state has the form

$$|\psi\rangle = \sum_{s\{\boldsymbol{m}\}} \psi_{s\boldsymbol{m}} |s\boldsymbol{m}\rangle. \tag{2.8}$$

Typically, an input state will not have a fixed s and instead will have support from many irreps. For example, the coherent state entering one input port and the vacuum state entering all the other $N-1$ input ports [4] can be expressed as

$$|\alpha\mathbf{0}\rangle = \exp\{-|\alpha|^2/2\} \sum_{Ns=0}^{\infty} \left(\alpha^{Ns}/\sqrt{(Ns)!}\right) |s\boldsymbol{s}\rangle \tag{2.9}$$

where $|s\boldsymbol{s}\rangle$ is a state of highest weight and the product Ns is an integer. On the other hand, the extension of the multiple Fock state input $|n, n, \ldots, n\rangle$ [11, 14, 15] for N fields corresponds to the input state $|s\mathbf{0}\rangle$.

The matrix elements of the unitary operator (2.3) correspond to the $SU(N)$ Wigner $\mathcal{D}$ functions

$$\langle s\boldsymbol{m}'|(\Upsilon)s\boldsymbol{m}\rangle = {}^{s}[\mathcal{I}(\Upsilon)]_{m_1m_2\ldots m_{N-1}}^{m_1'm_2'\ldots m_{N-1}'} \tag{2.10}$$

with

$$|(\Upsilon)s\boldsymbol{m}\rangle \equiv \mathcal{I}(\Upsilon)|s\boldsymbol{m}\rangle. \tag{2.11}$$

For $SU(2)$,

$$h_1 = 2J_z = M_1^1 - M_2^2 \tag{2.12}$$

and

$$M_2^1 = 2J_x \qquad M_2^1 = 2J_y \qquad J^2 = S(S+1). \tag{2.13}$$

The spectrum of the scalar parameter m is

$$m \in \{-s, -s+1, \ldots, s\}. \tag{2.14}$$

The weights can be viewed as equally spaced steps on a ladder. We can also use the more common $SU(2)$ notation for the weight basis as $|j\mu\rangle$, for $j = s$ and $\mu = m$.

For $SU(3)$, the weights are embedded in a two-dimensional space. For $s = n/3$, and n an integer, the weights are given by

$$\{\boldsymbol{m}\}_n = \{(n_1 - n_2, n_2 - n_3)\} \tag{2.15}$$

with $n_i \geqslant 0$ and $n_1 + n_2 + n_3 = n$. The cardinality of this set is

$$\mathcal{C}_s = (n+1)(n+2)/2 = (3s+1)(3s+2)/2. \tag{2.16}$$

For general $SU(N)$, Wigner $\mathcal{D}$ functions can be calculated by working with matrices in a specified irrep, but obtaining general explicit expressions for arbitrary s is challenging. We have developed Mathematica™ computer programs for calculating $SU(3)$ transformations for arbitrary representations following the methods of Rowe *et al* [13].

3. Vector phase representation

In considering rigorous bounds for extracting vector phase information, the first step is to determine the $SU(N)$ basis which is translated by the unitary interferometer transformation (2.3) for an arbitary phase-shift vector $\boldsymbol{\phi}$. This task is made easier by restricting attention to the category of interferometric experiments which we designate 'symmetric MQI' and define by equation (3.1) below. This nomenclature is distinct from that of symmetric multiports which correspond to linear $SU(N)$ transformers such that an incoming photon has an equal likelihood of exiting from each of the N output ports [16].

We consider the interferometer transformation (2.3) as a three-stage process. The first step, $\mathcal{D}$, transforms the input field into a new state and can be any $SU(N)$ interferometric transformation of the type (2.3).

In the second stage, an arbitrary phase transformation $\mathcal{P}(\boldsymbol{\phi}) \equiv \exp(\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{h})$ acts on the N paths. Finally, the field undergoes a mixing which is the reverse of $\mathcal{D}$. Thus, the unitary transformation for the symmetric MQI has the general form

$$\begin{aligned}\mathcal{I}(\boldsymbol{\phi}, \Upsilon) &= \mathcal{D}^\dagger(\Upsilon)\mathrm{e}^{\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{h}}\mathcal{D}(\Upsilon) = \mathcal{I}^\dagger(-\boldsymbol{\phi}, \Upsilon) \\ &= \exp\{\mathrm{i}\boldsymbol{\phi}\cdot[\mathrm{e}^{-\mathrm{i}\sum \Upsilon_l^k M_k^l}\boldsymbol{h}\mathrm{e}^{\mathrm{i}\sum \Upsilon_l^k M_k^l}]\}\end{aligned} \tag{3.1}$$

for some real $N \times N$ matrix Υ. This restriction to a symmetric MQI is quite reasonable. For example, the unitary operator for the (two-field) balanced Mach–Zehnder interferometer [10–12], depicted in figure 1(*a*), is

$$\begin{aligned}\mathcal{I}\left(\phi, \begin{bmatrix} 0 & \pi/4 \\ 0 & 0 \end{bmatrix}\right) &= \mathrm{e}^{-\mathrm{i}(\pi/4)M_2^1}\mathrm{e}^{-\mathrm{i}\phi h_1}\mathrm{e}^{\mathrm{i}(\pi/4)M_2^1} \\ &= \exp(2\mathrm{i}\phi M_2^1).\end{aligned} \tag{3.2}$$

The infinitesimal POVM for ideal vector phase measurement corresponding to the symmetric MQI (3.1), restricted to a particular irrep parametrized by s, of dimension $\mathcal{C}_s$, is

$$\mathrm{d}E_s(\boldsymbol{\theta}) = |(\Upsilon)s\boldsymbol{\theta}\rangle\langle(\Upsilon)s\boldsymbol{\theta}|\,\mathrm{d}\mu(\boldsymbol{\theta}) \tag{3.3}$$

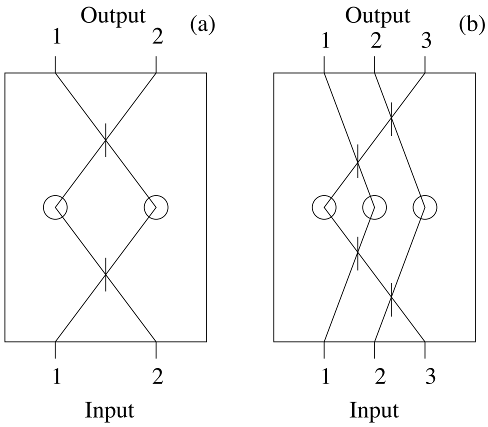

Figure 1. Symmetric (*a*) $SU(2)$ and (*b*) $SU(3)$ interferometers for measuring scalar and two-dimensional vector phase, respectively. The input fields are at the bottom and the output fields at the top of the diagram. The vertical lines represent 50/50 beam splitters, and the circles represent phase shifters which induce an arbitrary phase shift of ϕ_n in arm n.

where

$$|(\Upsilon)s\boldsymbol{\theta}\rangle = \mathcal{C}_s^{-1/2}\sum_{\{\boldsymbol{m}\}}\exp\{\mathrm{i}[(n_1-n_2)\theta_1+(n_2-n_3)\theta_2]\}\mathcal{D}^\dagger(\Upsilon)|s\boldsymbol{m}\rangle \tag{3.4}$$

is an $SU(N)$ VPS,

$$\mathrm{d}\mu(\boldsymbol{\theta}) = \frac{\mathcal{C}_s}{(2\pi)^{N-1}}\,\mathrm{d}^{N-1}\boldsymbol{\theta} \tag{3.5}$$

and $\{\boldsymbol{m}\}$ is the set of weights for the symmetric representation of $SU(N)$ parametrized by s. We emphasise that the set of phase states (3.4) is not the dual basis to the weight basis of $\boldsymbol{h}$ but rather dual to the 'rotated' weight basis $|(\Upsilon)s\boldsymbol{m}\rangle$. This $SU(N)$ 'rotation' is essential to guarantee that the VPS is indeed translated by the unitary interferometer transformation operator (3.1).

The phase state (3.4) reduces to the rotated $SU(2)$ phase state for $N=2$ [11, 17]. An orthonormal basis for the Hilbert space can be constructed with $\mathcal{C}_s$ orthonormal phase states. The matrix element connecting the weight basis and (3.4) is given by

$$\langle(\Upsilon)s\boldsymbol{\theta}|s\boldsymbol{m}\rangle = \mathcal{C}_s^{-1/2}\sum_{\{\boldsymbol{m}'\}}\mathrm{e}^{-\mathrm{i}\boldsymbol{m}'\cdot\boldsymbol{\theta}}\langle s\boldsymbol{m}'|\mathcal{D}(\Upsilon)|s\boldsymbol{m}\rangle. \tag{3.6}$$

The phase distribution for an arbitrary input state (2.8) is given by

$$\mathrm{d}P_s(\theta) = \frac{\mathcal{C}_s}{(2\pi)^{N-1}}\left|\sum_{\{\boldsymbol{m}\}}\psi_{s\boldsymbol{m}}\langle(\Upsilon)s\boldsymbol{\theta}|s\boldsymbol{m}\rangle\right|^2\mathrm{d}^{N-1}\boldsymbol{\theta}. \tag{3.7}$$

That is, the vector phase distribution is obtained from the phase representation of the state by squaring the modulus of the overlap between the state and a vector phase state, and then normalizing. In general, the coefficients $\{\psi_{s\boldsymbol{m}}\}$ are calculated via the conversion formula from the Fock basis to the weight basis. The phase distribution for the output state is given by $P_s(\boldsymbol{\theta}|\boldsymbol{\phi}) = P_s(\boldsymbol{\theta}-\boldsymbol{\phi})$. This distribution can be used then to determine the ultimate bounds on estimating the induced vector phase shift $\boldsymbol{\phi}$.

One method for determining bounds to extracting phase-shift information is by the Fisher information method [8]. However, for vector phase, the Fisher information must be replaced by the Fisher information matrix, which is given [6] by

$$\boldsymbol{F}_s = \frac{\mathcal{C}_s}{(2\pi)^{N-1}}\int \mathrm{d}^{N-1}\boldsymbol{\theta}\; P_s(\boldsymbol{\theta}|\boldsymbol{\phi})[\nabla\ln P_s(\boldsymbol{\theta}|\boldsymbol{\phi})]\times[\nabla\ln P_s(\boldsymbol{\theta}|\boldsymbol{\phi})]. \tag{3.8}$$

In equation (3.8), $\times$ indicates the outer product of vectors and ∇ is the $(N-1)$-dimensional gradient with respect to $\boldsymbol{\theta}$. For $\boldsymbol{F}^{-1}$ the inverse of the Fisher information matrix (3.8) and $\delta\boldsymbol{\phi}$ the uncertainty in the estimate of $\boldsymbol{\phi}$, the products of uncertainties are bounded from below according to the requirement that the matrix

$$\delta\boldsymbol{\phi} \times \delta\boldsymbol{\phi} - \boldsymbol{F}^{-1} \tag{3.9}$$

is positive definite and $\times$ once again denotes the outer product of vectors.

Expression (3.8) can be simplified by noting that the phase distribution is translated by the interferometer transformation. Hence, $P_s(\boldsymbol{\theta}|\boldsymbol{\phi}) = P_s(\boldsymbol{\theta} - \boldsymbol{\phi})$. Consequently, the Fisher information matrix is independent of $\boldsymbol{\phi}$. That the Fisher information is independent of the applied phase shift in the interferometer should not be surprising. The choice of the vector phase POVM is designed to produce this result.

In contrast, phase information which is obtained via particle counting [4,18] corresponds to phase-shift estimates via weight-basis distributions. In analyses of limits to phase information extraction via particle counting, the weight basis is paramount, but our objective has been to obtain equation (3.8), which establishes the absolute, *in principle*, bound to extracting phase information from the system with or without particle counting methods.

4. The symmetric $SU(3)$ interferometer

An example of symmetric three-field MQI is shown in figure 1(*b*). The three input beams at the bottom of the diagram are directed into two beam splitters where they are mixed in a symmetric way, then subjected to phase shifts (each beam k, for $k = 1, 2, 3$, is subjected to a phase shift ϕ_k, respectively) and finally directed to two more beam splitters before exiting the system. The unitary transformation for the three-beam interferometer in figure 1(*b*) is given by

$$\mathcal{I}(\boldsymbol{\phi}) = \mathrm{e}^{-\mathrm{i}\pi M_3^2/4}\mathrm{e}^{-\mathrm{i}\pi M_2^1/4}\mathrm{e}^{\mathrm{i}\boldsymbol{\varphi}\cdot\boldsymbol{h}}\mathrm{e}^{\mathrm{i}\pi M_2^1/4}\mathrm{e}^{\mathrm{i}\pi M_3^2/4} \tag{4.1}$$

with

$$\boldsymbol{\varphi} = (\phi_1 - \phi_2, \phi_1/2 + \phi_2/2 - \phi_3) \tag{4.2}$$

up to a global phase factor $\Phi = \phi_1 + \phi_2 + \phi_3$.

A detailed study of the $SU(3)$ interferometer helps to clarify the case of $SU(N)$ interferometry. Let us rewrite the interferometer transformation (4.1) using the notation of equation (3.1)

$$\mathcal{I}(\boldsymbol{\phi}) = \mathcal{D}^\dagger \mathrm{e}^{\mathrm{i}\boldsymbol{\varphi}\cdot\boldsymbol{h}}\mathcal{D} \tag{4.3}$$

for

$$\mathcal{D} = \mathrm{e}^{\mathrm{i}\pi M_2^1/4}\mathrm{e}^{\mathrm{i}\pi M_3^2/4}. \tag{4.4}$$

The desired VPS is

$$|(\Upsilon)s\boldsymbol{\theta}\rangle = \frac{2}{\sqrt{(3s+1)(3s+2)}}\sum_{\boldsymbol{m}} \mathrm{e}^{\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\theta}}\mathcal{D}^\dagger(\Upsilon)|s\boldsymbol{m}\rangle \tag{4.5}$$

and a pure state $|\psi\rangle$ with fixed s has a phase distribution

$$\begin{aligned} \mathrm{d}P_s(\boldsymbol{\theta}) &= \frac{(3s+1)(3s+2)}{2(2\pi)^2}|\langle(\Upsilon)s\boldsymbol{\theta}|\psi\rangle|^2\,\mathrm{d}^{N-1}\boldsymbol{\theta} \\ &= \left|\sum_{\boldsymbol{m}'} \mathrm{e}^{-\mathrm{i}\boldsymbol{m}'\cdot\boldsymbol{\theta}}\langle s\boldsymbol{m}'|\mathcal{D}(\Upsilon)|s\boldsymbol{m}\rangle\right|^2 \frac{\mathrm{d}^{N-1}\boldsymbol{\theta}}{2\pi^2}. \end{aligned} \tag{4.6}$$

We consider specifically two input states. For both states, s is restricted to being an integer. The first is the highest-weight state $|ss\rangle$, which has all particles entering just one of the three input ports. The second state under consideration is the balanced state $|s\mathbf{0}\rangle$, which has an equal number of particles entering each input port.

The algorithm for calculating matrix elements of $\mathcal{D}$ is provided in [13]. A general $SU(3)$ transformation is written as

$$\mathcal{D}(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_3) = \mathcal{R}_2^3(\alpha_1, \beta_1, \gamma_1)\mathcal{R}_1^2(\alpha_2, \beta_2, \alpha_2)\mathcal{R}_2^3(\alpha_3, \beta_3, \gamma_3) \tag{4.7}$$

with

$$\mathcal{R}_i^j(\alpha, \beta, \gamma) = \mathrm{e}^{\mathrm{i}\alpha(M_i^i - M_j^j)/2}\mathrm{e}^{\mathrm{i}\beta M_j^i/2}\mathrm{e}^{\mathrm{i}\gamma(M_i^i - M_j^j)/2} \tag{4.8}$$

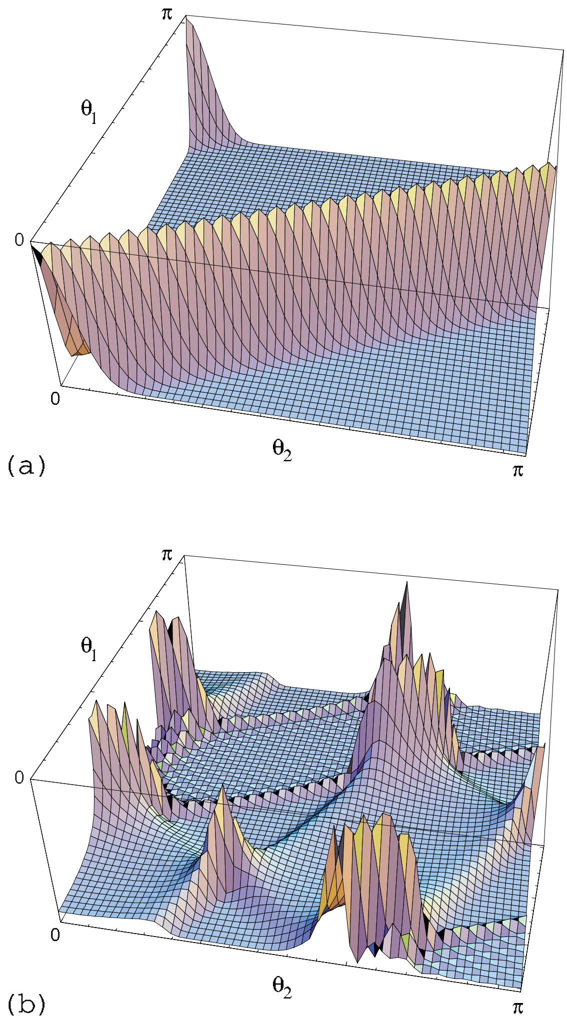

Figure 2. Vector phase distributions P for (a) the highest-weight state $|ss\rangle$ and (b) the balanced state for $s = 12$ (36 photons). For angles outside of the range shown, the vector phase distribution is given by $P(\theta_1 + \pi, \theta_2 + \pi)$ and $P(\theta_1 + 2\pi, \theta_2 + 2\pi) = P(\theta_1, \theta_2)$.

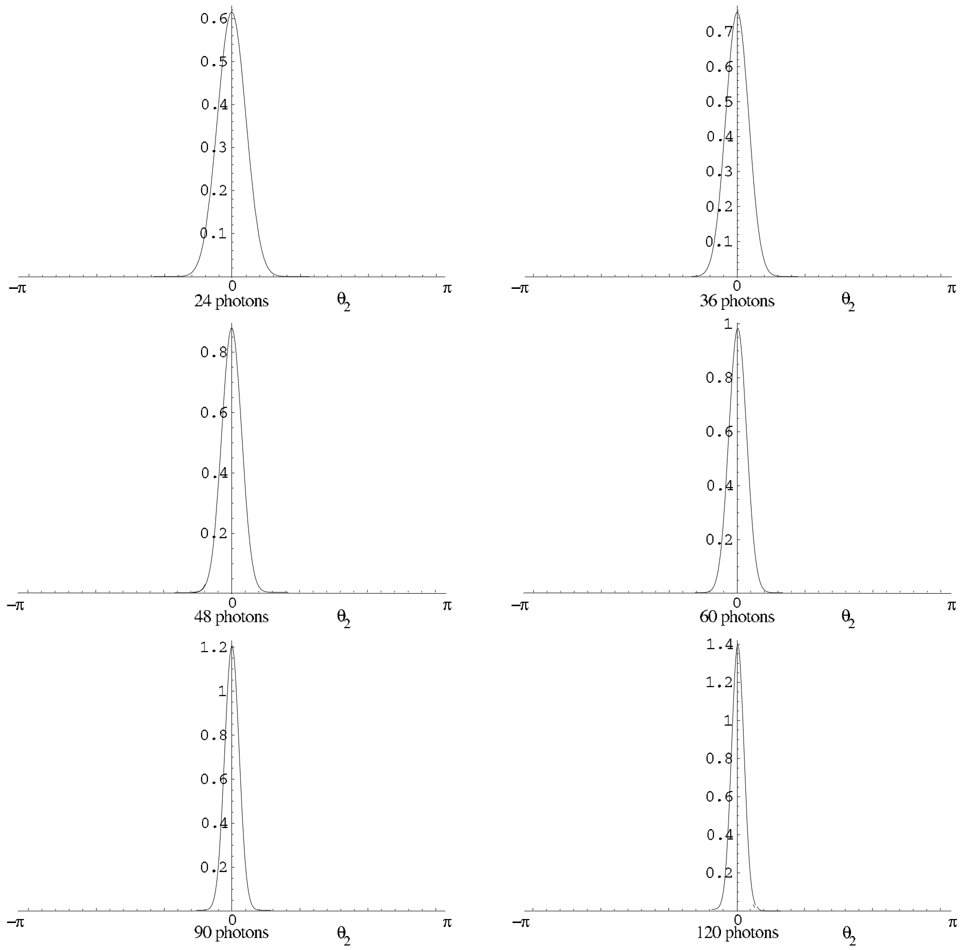

Figure 3. Vector phase distributions for the highest-weight state $|ss\rangle$ for $\theta_1 = 0$ and $3s \in \{24, 36, 48, 60, 90, 120\}$ photons.

the usual factorization of an $SU(2)$ transformation. For the transformation (4.4) the general $SU(3)$ transformation (4.7) assumes the simple form

$$\mathcal{D} = \mathcal{D}(0, 0, 0, 0, \pi/2, 0, \pi/2, 0). \tag{4.9}$$

Inserting expression (4.9) for $\mathcal{D}$ into equation (4.6) provides the necessary expression for computing the vector phase distribution over the two free parameters $\boldsymbol{\theta} = (\theta_1, \theta_2)$.

The phase distributions have been calculated using Mathematica$^{\mathrm{TM}}$ and are plotted in figures 2–4. Calculations are performed for the highest-weight state $|ss\rangle$ and the balanced state $|s\mathbf{0}\rangle$. Surface plots are presented in figure 2 for both the highest-weight state and the balanced state with $s = 12$. This choice of s corresponds to a total of $3s = 36$ photons in the interferometer. The highest-weight state is an $SU(3)$ coherent state [19], and the surface plot is the two-dimensional phase distribution for this state. The importance of these phase distributions is that the output state has the same phase distribution as that shown in figure 2 except for a translation in the (θ_1, θ_2) plane.

In figures 3 and 4 phase distributions are plotted for the highest-weight state and balanced state, respectively. The sequence of plots are slices of the surface plots corresponding to

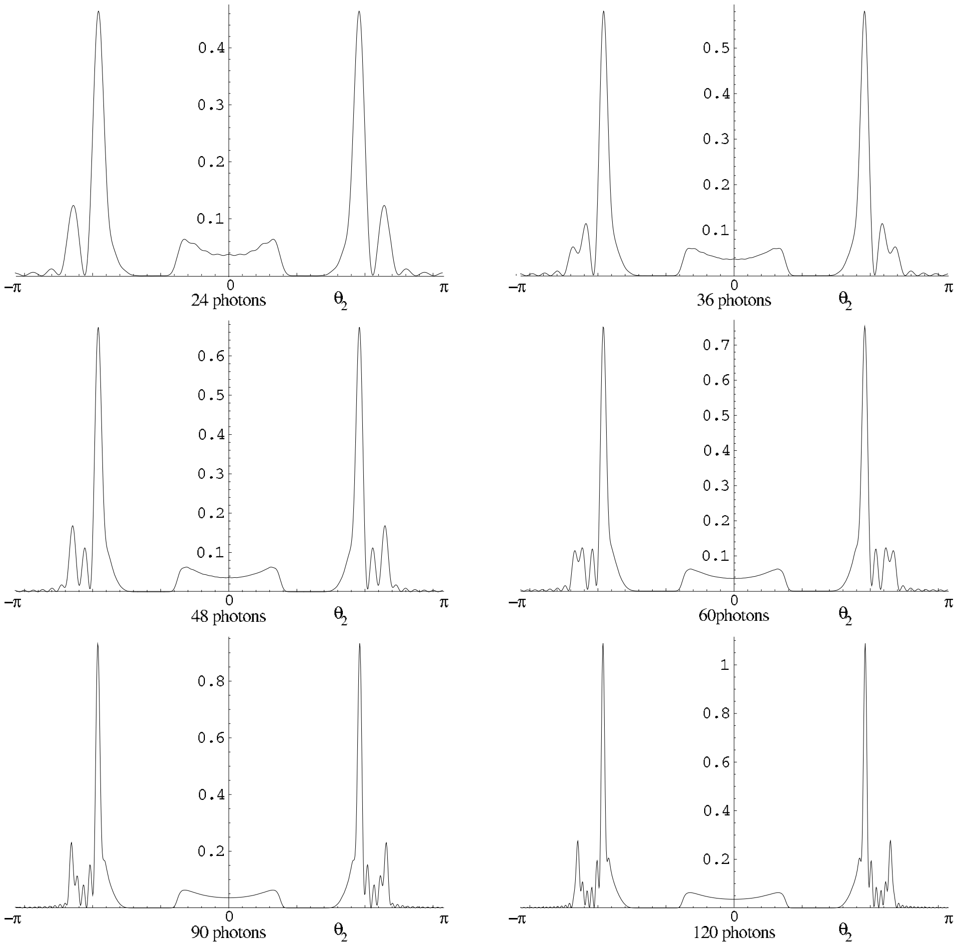

Figure 4. Vector phase distributions for the balanced state $|s\mathbf{0}\rangle$ for $\theta_1 = 0$ and $3s \in \{24, 36, 48, 60, 90, 120\}$ photons.

fixing θ_1 for photon number $3s$ increasing from 24 to 120. The major peak is used for both the highest-weight state and the balanced state to infer the imposed phase shift. One way to analyse the precision of phase-shift estimation is to consider the width of the major peak along slices of the surface plot. This provides information about the scaling of the precision with respect to classes of phase shifts along one dimension in the two-dimensional space (θ_1, θ_2).

The scaling of the width of the phase distribution for the $SU(3)$ coherent state, or highest-weight state, in figure 3 approaches $1/\sqrt{s}$. This scaling is consistent with the scaling of the phase precision for $SU(2)$ coherent states [11]. Slices of the phase distribution for the balanced state, depicted in figure 4, exhibit a scaling of width which is superior to $1/\sqrt{s}$ (but not at the $SU(2)$ scaling of $1/s$). For $SU(3)$ interferometry, and the configuration depicted in figure 1(*b*), the $SU(3)$ balanced state provides a superior phase-shift estimate in terms of scaling with respect to input photon number.

5. Conclusions

In conclusion, we have developed the $SU(N)$ VPS. The rotated $SU(N)$ VPS has proven useful in estimating bounds to phase-shift measurements. Specifically, the $SU(3)$ vector phase distribution has been used to observe a $1/\sqrt{s}$ scaling for the highest-weight (or coherent) state and a superior scaling of phase-shift estimate precision for the balanced input state. Scaling of phase-shift estimate precision was studied in the context of symmetric MQI, for which phase-shift measurements and bounds to estimation provide particularly straightforward mathematical results. We have also clarified the use of $SU(N)$ vector phase distributions in the context of the Fisher information matrix for establishing bounds on multiparameter estimation.

We have relied on numerical simulations to study $SU(3)$ two-dimensional phase distributions and the scaling of the precision of the phase-shift estimates with respect to photon number $3s$. An alternative approach is to employ asymptotic methods for $SU(3)$ transformations [22]. Asymptotic $SU(3)$ expressions would simplify the study of measurement limits for $SU(3)$ interferometry, analogous to the asymptotic approach to studying measurement limits for $SU(2)$ interferometry [11]. We are currently developing the application of asymptotic methods for $SU(3)$ interferometry.

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